What are… Pfaffians and Perfect Matchings?

1. Matchings

Given a graph $G(V, E)$ with vertex set V and edge set E, a matching is a subset $M \subseteq E$ such that no two edges in M share a common vertex.

A perfect matching is a matching in which every vertex of G is met by an edge.

We wish to develop a determinantial formula for the generating function of perfect matchings in a graph.

2. The Pfaffian and Skew Symmetric Matrices

In 1882 Thomas Muir proved that the determinant of a skew matrix is the square of a polynomial, and noted that this polynomial was the Pfaffian Caley mentioned in 1852 in his essay "On the theory of permutants." A formal definition for the determinant of an $n \times n$ matrix is given by

$$det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

For a $2n \times 2n$ skew matrix (each entry is the opposite of its conjugate) a formal definition of the Pfaffian may be adapted from that of the determinant and is given by

$$pf(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} sgn(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)}$$

However, most of these terms are redundant for our purposes. For example, if we swap $(2i-1)$with $(2j-1)$ and $(2i)$ with $(2j)$ we effectively permute the factors $a_{i,j}$. Reordering the factors $a_{i,j}$ does not change the sign of σ because it is an even number of swaps from one permutation σ to another. There are $n!$ such permutations of the factors $a_{i,j}$. Additionally, the order of the indices of each $a_{i,j}$ may be reversed. This is a single swap in the permutation, so the sign of σ switches. However, since the matrix is skew $a_{i,j} = -a_{j,i}$ and thus the sign of the term as a whole remains constant. There are 2^n ways to order all the indices together. If we order the factors in each term and the indices of each factor lexigraphically, we obtain $2^n n!$ instances of each term, so the following definition of the Pfaffian is equivalent.

Let Π be the set of all partitions of $\{1, 2, \dots, 2n\}$ into pairs without regard to order. An element alpha in Π can be written in lexigraphical order as

$$\propto = \{(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)\}$$

with $i_k < j_k$ and $i_1 < i_2 < \cdots < i_n$. Let

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & \dots & 2n-1 & 2n \\ i_1 & j_1 & i_2 & j_2 & \dots & i_n & j_n \end{bmatrix}$$

be the corresponding permutation.

Define

$$A_{\propto} = sgn(\pi)a_{i_1,j_1}a_{i_2,j_2}\dots a_{i_n,j_n}.$$

Then the Pfaffian of A is given by

$$pf(A) = \sum_{\propto\in\Pi} A_{\propto}.$$

If $\{1, 2, \dots, 2n\}$ is the set of vertices in a graph G and the weights $a_{i,j}$ are the weights of the edges connecting vertices i and j (with $a_{i,j} = 0$ for non-edges) then it is clear that the non-zero terms of the Pfaffian enumerate all the perfect matchings in G (ignoring signs).

What is left to do, then, is to find a suitable matrix constructed from our graph G whose determinant is the squared Pfaffian of G.

3. Adjacency Matrices and the Tutte Matrix

A natural matrix representation of a graph is its adjacency matrix. For a graph $G(V, E)$ we number the vertices and define an adjacency matrix

$$A_{ij} = \begin{cases} 1 \; if \; v_i v_j \in E \\ 0 \; otherwise \end{cases}$$

Since each entry is equal to its conjugate, however, the adjacency matrix is symmetric, not skew. The skew counterpart to the adjacency matrix is the directed adjacency matrix. We assign an orientation to the edges of G and define the directed adjacency matrix

$$D_{ij} = \begin{cases} 1 \; if \; v_i v_j \in E \\ -1 \; if \; v_j v_i \in E \\ 0 \; otherwise \end{cases}$$

We see that this is equivalent to assigning a weight of 1 to each directed edge, and in fact there is a large set of graphs (containing the planar graphs) called Pfaffian graphs for which an orientation exists such that the Pfaffian of D is the number of perfect matchings.

To enumerate all the perfect matchings we simply assign indeterminates to the weights of the edges, producing the Tutte matrix

$$T_{ij} = \begin{cases} x_{ij} \; if \; v_i v_j \in E \\ -x_{ji} \; if \; v_j v_i \in E \\ 0 \; otherwise \end{cases}$$

Thus, the positive square root of $det(T)$ is the Pfaffian of G, which enumerates all of its perfect matchings.

4. Brief History

Johann Friedric Pfaff (1765-1825) was a German mathematician best known for his work in integral calculus and systems of partial differential equations of the first order (now called Pfaffian systems). His notable students included Carl Friedrich Gauss and August Ferdinand Möbius.

In the field of differential equations, Pfaff's Problem is, given a manifold (space locally like R^n) M and a set S of differential forms (e.g. $f(x)dx$), to find all submanifolds (subsets of M themselves manifolds) on which the pullback (linear map from 1-forms of N to 1-forms of M is the pullback of $\varphi: M \to N$) of any form in S is identically zero.

The term Pfaffian was introduced by Caley in 1852 to describe certain functions appearing in the solution to Pfaff's Problem.

In 1882 Thomas Muir proved that the positive square root of the determinant of a skew symmetric matrix is a Pfaffian.

The Pfaffian has been studied extensively for its applications in (among other fields) computer science, where perfect matchings are of particular interest. It is often considered more fundamental than the determinant due to the identity

$$\det(M) = (-1)^{-\frac{n(n-1)}{2}} pf \begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix}$$

5. References

Cayley, Arthur (1852). "On the theory of permutants". Cambridge and Dublin Mathematical Journal VII: 40–51.
http://www.archive.org/stream/collectedmathema02cayluoft#page/16/mode/2up

Muir, Thomas (1882). "A Treatise on the Theory of Determinants". Macmillan and Co.. 195-205.
http://books.google.com/books?id=pk4DAAAAQAAJ&dq=thomas+muir+treatise+on+the+theory+of+determinant&psp=1

Rote, Günter (2001). "Division-Free Algorithms for the Determinant and the Pfaffian: Algebraic and Combinatorial Approaches". Computational Discrete Mathematics: 119-135. Springer Berlin / Heidelberg.
http://www.springerlink.com/content/59ktw519u18a63h7/

Tutte, W.T. (April 1947). "The factorization of linear graphs". J. London Math. Soc. 22 (2): 107–111.
http://jlms.oxfordjournals.org/cgi/reprint/s1-22/2/107.pdf